

CONSTRUCTION OF SEPARATELY CONTINUOUS FUNCTIONS OF n VARIABLES WITH GIVEN RESTRICTION

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ABSTRACT. It is solved the problem on construction of separately continuous functions on product of n topological spaces with given restriction. In particular, it is shown that for every topological space X and $n - 1$ Baire class function $g : X \rightarrow \mathbb{R}$ there exists a separately continuous function $f : X^n \rightarrow \mathbb{R}$ such that $f(x, x, \dots, x) = g(x)$ for every $x \in X$.

1. INTRODUCTION

For every set X and an integer $n \geq 2$ the mapping $d_n : X \rightarrow X^n$, $d_n(x) = (x, \dots, x)$, is called *diagonal mapping*. The set $\Delta_n = d_n(X)$ is called *the diagonal of the space X* , and the composition $g = f \circ d_n : X \rightarrow Y$ is called *the diagonal of mapping $f : X^n \rightarrow Y$* .

Let X be a topological space. A mapping $f : X \rightarrow \mathbb{R}$ is called *a function of the first Baire class* if there exists a sequence $(f_n)_{n=1}^\infty$ of continuous function $f_n : X \rightarrow \mathbb{R}$ which converges to f pointwise on X , i.e. $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for every $x \in X$. Let $2 \leq \alpha < \omega_1$. A mapping $f : X \rightarrow \mathbb{R}$ is called *a function of α -th Baire class* if there exists a sequence (f_n) of at most Baire class $< \alpha$ functions $f_n : X \rightarrow \mathbb{R}$ which converges to f pointwise on X .

R. Baire shows in [1] that the diagonals of separately continuous functions of two real variables (i.e. functions which are continuous with respect each variable) are exactly the first Baire functions. A. Lebesgue proved in [2] that every separately continuous function of n real variables is a $(n - 1)$ -th Baire class function, in particular, its diagonal is a function of the same class. Conversely, it was shown in [3,4] that every real function of $(n - 1)$ -th Baire class is the diagonal of a separately continuous function of n real variables.

Beginning from the second half of the 20-th century many mathematicians (see [5-10]) studied actively Baire classification of separately continuous functions and their analogs. Note that W. Rudin firstly used a partition of the unit for the establishment of the belonging to the first Baire class of a separately continuous function defined on the product of metrizable and topological spaces and valued in a locally convex space. A development of the Rudin's method on the case of nonmetrizable spaces leads to the appearance of the following notion [11].

Topological space X is called *a PP-space* if there exist a sequence of locally finite covers $(U(n, i) : i \in I_n)$ which of functionally open in X sets $U(n, i)$ and sequence of families $(x(n, i) : i \in I_n)$ of points $x(n, i) \in X$ such that for every $x \in X$ and

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neighborhood U of x there exists $n_0 \in \mathbb{N}$ such that $x(n, i) \in U$ for every $n \geq n_0$ and $i \in I_n$ with $x \in U(n, i)$.

Class of σ -spaces is quite wide. It contains σ -metrizable paracompact spaces, topological vector spaces, which can be presented as the union of an increasing sequence of metrizable subspaces, Nemytski plane and Sorgenfrey line. It follows from [12] (see also [8, Theorem 3.14]) that for every $n \geq 2$ and PP -space X every separately continuous function $f : X^n \rightarrow \mathbb{R}$ is a function of the $(n - 1)$ -th Baire class and, in particular, its diagonal is a function of $(n - 1)$ -th Baire class.

The inverse problem on the construction of separately continuous function with the given diagonal was overlooked by mathematicians during a long time. This investigation was recrudescenced by V.Maslyuchenko. The most general result for separately continuous functions of n variables was obtained in [13]. It was obtained in [13] (see also [8, Theorem 3.24]) that for every function g of $(n - 1)$ -th Baire class, which defined on a topological space X with a normal n -th power and a G_δ -diagonal Δ_n there exists a separately continuous function $f : X^n \rightarrow \mathbb{R}$ with the diagonal g .

On other hand, in the investigations of separately continuous functions $f : X \times Y \rightarrow \mathbf{R}$ defined on the product of topological spaces X and Y the following two topologies naturally arise (see [14]): the separately continuous topology σ (the weakest topology with respect to which all functions f are continuous) and the cross-topology γ (it consists of all sets G for which all x -sections $G^x = \{y \in Y : (x, y) \in G\}$ and y -sections $G_y = \{x \in X : (x, y) \in G\}$ are open in Y and X respectively). The separately continuous topology σ and the cross-topology γ on the product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces X_1, X_2, \dots, X_n . Since the diagonal $\Delta = \{(x, x) : x \in \mathbf{R}\}$ is a closed discrete set in (\mathbf{R}^2, σ) or in (\mathbf{R}^2, γ) and not every function defined on Δ can be extended to a separately continuous function on \mathbf{R}^2 , even for $X = Y = \mathbf{R}$ the topologies σ and γ are not normal (moreover, γ is not regular [14, 15]). Thus, the construction separately continuous functions with the given diagonal is a partial case of more general problem: to establish for which subsets E of a product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces X_1, X_2, \dots, X_n and σ -continuous or γ -continuous function $g : E \rightarrow \mathbf{R}$ there exists a separately continuous function $f : X \times Y \rightarrow \mathbf{R}$ for which the restriction $f|_E$ coincides with g .

This question for functions of two variables was study in [16]. It was obtained in [16] that for every topological space X and a function $g : X \rightarrow \mathbb{R}$ of the first Baire class there exists a separately continuous function $f : X^2 \rightarrow \mathbb{R}$ with the diagonal g .

In this paper analogously as in [16] we solve the problem on the construction of separately continuous function $f : X_1 \times \cdots \times X_n \rightarrow \mathbb{R}$ with given restriction on a special type set $E \subseteq X_1 \times \cdots \times X_n$. In particular, we obtain that the for a topological space X and a function $g : X \rightarrow \mathbb{R}$ of $(n - 1)$ -th Baire class there exists a separately continuous function $f : X^n \rightarrow \mathbb{R}$ with the diagonal g .

2. NOTIONS AND AUXILIARY STATEMENT

A set $A \subseteq X$ has the extension property in a topological space X , if every continuous function $g : A \rightarrow [0, 1]$ can be extended to a continuous function $f : X \rightarrow [0, 1]$. According to Tietze-Uryson theorem [17, p.116], every closed set in a normal space has the extension property.

For a mapping $f : X \rightarrow Y$ and a set $A \subseteq X$ by $f|_A$ we denote the restriction of f on A .

A set A in a topological space X is called *functionally closed* if there exists a continuous function $f : X \rightarrow [0, 1]$ such that $A = f^{-1}(0)$.

A topological space X is called *pseudocompact* if every continuous on X function is bounded.

A set E in the product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces X_1, X_2, \dots, X_n is called *projectively homeomorphic* if for every $1 \leq i \leq n$ the projection $p_i : E \rightarrow X_i$, $p_i(x_1, x_2, \dots, x_n) = x_i$, is a homeomorphic embedding.

A set E in the product $X_1 \times X_2 \times \cdots \times X_n$ is called *projectively injective* if $x_1 \neq y_1, x_2 \neq y_2, \dots, x_n \neq y_n$ for every distinct points $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in E$, i.e. all projections of the set E on axis X_i are injective and *locally projectively injective* if for every $x \in X_1 \times X_2 \times \cdots \times X_n$ there exists a neighborhood W of x such that the set $E \cap W$ is projectively injective.

For a function $f : X \rightarrow \mathbb{R}$ by $\text{supp} f$ we denote the set $\{x \in X : f(x) \neq 0\}$.

Proposition 2.1. *Let A be a functionally closed set which has the extension property in a topological space X , $1 \leq \alpha < \omega_1$ and $g : A \rightarrow \mathbb{R}$ be a function of α -th Baire class. Then the function $f : X \rightarrow \mathbb{R}$ which defined by: $f(x) = g(x)$ for every $x \in A$ and $f(x) = 0$ for every $x \in X \setminus A$, is a function of α -th Baire class too.*

Proof. It follows from the definition of function of α -th Baire class that it enough to prove the statement for $\alpha = 1$.

Let $\alpha = 1$ and $(g_n)_{n=1}^\infty$ be a sequence of continuous functions $g_n : A \rightarrow [-n, n]$, which pointwise converges to the function g . Since the set A has the extension property in the topological space X , there exists a sequence $(f_n)_{n=1}^\infty$ of continuous function $f_n : X \rightarrow \mathbb{R}$ such that $f_n|_A = g_n$.

We take a continuous function $\varphi : X \rightarrow [0, 1]$ such that $A = \varphi^{-1}(0)$ and for every $n \in \mathbb{N}$ and $x \in X$ we put $\varphi_n(x) = 1 - \min\{1, n\varphi(x)\}$. Clearly that all functions φ_n are continuous on X , $A = \varphi_n^{-1}(1)$ and for every $x \in X \setminus A$ there exists an integer $m \in \mathbb{N}$ such that $\varphi_n(x) = 0$ for every $n \geq m$. Then the sequence of continuous functions $f_n \cdot \varphi_n$ pointwise converges to the function f . \square

3. MAIN RESULTS

Theorem 3.1. *Let E be a projectively homeomorphic set in the product $X_1 \times \cdots \times X_n$ of topological spaces X_1, \dots, X_n , moreover the projections E_1, \dots, E_n of the set E have the extension property in the spaces X_1, \dots, X_n respectively and $g : E \rightarrow \mathbb{R}$ be a function of $(n-1)$ -th Baire class. Then if E is pseudocompact or all sets E_1, \dots, E_n are functionally closed in X_1, \dots, X_n , then there exists a separately continuous function $f : X_1 \times \cdots \times X_n \rightarrow \mathbb{R}$ such that $f|_E = g$.*

Proof. Firstly we consider the case of functionally closed sets E_1, \dots, E_n in X_1, \dots, X_n respectively.

Let

$$f_0^{(1)} : X_1 \rightarrow [0, 1], \dots, f_0^{(n)} : X_n \rightarrow [0, 1]$$

be continuous functions such that

$$E_i = (f_0^{(i)})^{-1}(0)$$

for every $1 \leq i \leq n$. For every $i = 1, \dots, n$ and $x = (x_1, \dots, x_n) \in E$ we put $h_i(x_i) = x$. Clearly that h_i is a homeomorphism of the set E_i on the set E . Since

g is a function of $(n-1)$ -th Baire class on E , there exists a family $(g_{k_1, \dots, k_{n-1}} : k_1, \dots, k_n \in \mathbb{N})$ of continuous functions $g_{k_1, \dots, k_{n-1}} : E \rightarrow [-k_{n-1}, k_{n-1}]$ such that

$$g(x) = \lim_{k_1 \rightarrow \infty} \lim_{k_2 \rightarrow \infty} \dots \lim_{k_{n-1} \rightarrow \infty} g_{k_1, k_2, \dots, k_{n-1}}(x)$$

for every $x \in E$. For every $k_1, \dots, k_{n-1} \in \mathbb{N}$ and $1 \leq i \leq n$ by $g_{k_1, \dots, k_{n-1}}^{(i)}$ we denote the continuous function

$$g_{k_1, \dots, k_{n-1}}^{(i)} : E_i \rightarrow [-k_{n-1}, k_{n-1}], \quad g_{k_1, \dots, k_{n-1}}^{(i)}(x_i) = g_{k_1, \dots, k_{n-1}}(h_i(x_i))$$

and choose a continuous function $f_{k_1, \dots, k_{n-1}}^{(i)} : X_i \rightarrow \mathbb{R}$ such that

$$f_{k_1, \dots, k_{n-1}}^{(i)}|_{E_i} = g_{k_1, \dots, k_{n-1}}^{(i)}.$$

We put $S = \{0\} \cup \mathbb{N}^{n-1}$. Further for every $s = (k_1, \dots, k_{n-1}) \in S$ and $1 \leq i \leq n$ the functions $f_{k_1, \dots, k_{n-1}}^{(i)}$, $g_{k_1, \dots, k_{n-1}}^{(i)}$ and $g_{k_1, \dots, k_{n-1}}$ we denote by $f_s^{(i)}$, $g_s^{(i)}$ and g_s .

We consider the continuous mappings

$$\varphi_i = \Delta_{s \in S} f_s^{(i)} : X_i \rightarrow \mathbb{R}^S, \quad \varphi_i(x_i) = (f_s^{(i)}(x_i))_{s \in S}.$$

We denote $Z = \bigcup_{i=1}^n \varphi_i(X_i)$. Note that Z is a metrizable space and for every $1 \leq$

$i \leq n$, $s \in \mathbb{N}^{n-1}$ and $x = (x_1, \dots, x_n) \in E$ we have $f_s^{(i)}(x_i) = g_s^{(i)}(x_i) = g_s(x)$. Moreover for every $1 \leq i \leq n$ the point x_i from X_i belongs to the set E_i if and only if $\varphi_i(x_i)(0) = f_0^{(i)}(x_i) = 0$. Therefore $\varphi_1(x_1) = \varphi_2(x_2) = \dots = \varphi_n(x_n)$ for every point $(x_1, x_2, \dots, x_n) \in E$, besides the set $A = \varphi_1(E_1) = \dots = \varphi_n(E_n)$ is functionally closed in Z .

We consider the function $\tilde{g} : A \rightarrow \mathbb{R}$, $\tilde{g}(z) = g(h_1(x_1))$, where $x_1 \in E_1$ and $z = \varphi_1(x_1)$. Note that for every $x_1, y_1 \in E_1$ the equality $\varphi_1(x_1) = \varphi_1(y_1)$ implies that $g_s^{(1)}(x_1) = g_s^{(1)}(y_1)$ for every $s \in S$. Then $g_s(h_1(x_1)) = g_s(h_1(y_1))$ for every $s \in \mathbb{N}^{n-1}$ and $g(h_1(x_1)) = g(h_1(y_1))$. Thus, the definition of the function \tilde{g} is correct.

For every $s = (k_1, \dots, k_{n-1}) \in S$ the function $\tilde{f}_s : Z \rightarrow \mathbb{R}$, $\tilde{f}_s(z) = z(s)$, is continuous. Then for every $z \in A$ we choose $x_1 \in E_1$ such that $\varphi_1(x_1) = z$ and obtain

$$\tilde{f}_s(z) = z(s) = f_s^{(1)}(x_1) = g_s^{(1)}(x_1) = g_s(h_1(x_1)).$$

Therefore,

$$\lim_{k_1 \rightarrow \infty} \dots \lim_{k_{n-1} \rightarrow \infty} \tilde{f}_s(z) = \lim_{k_1 \rightarrow \infty} \dots \lim_{k_{n-1} \rightarrow \infty} g_s(h_1(x_1)) = g(h_1(x_1)) = \tilde{g}(z).$$

Thus, the function \tilde{g} is a function of $(n-1)$ -th Baire class on A . According to Proposition 2.1, if we put $\tilde{g}(z) = 0$ for every $z \in Z \setminus A$, then we obtain the function \tilde{g} of $(n-1)$ -th Baire class on Z . It follows from [13, Theorem 2] that there exists a separately continuous function $\tilde{f} : Z^n \rightarrow \mathbb{R}$ with the diagonal \tilde{g} .

Consider the separately continuous function

$$f : X_1 \times \dots \times X_n \rightarrow \mathbb{R}, \quad f(x_1, \dots, x_n) = \tilde{f}(\varphi_1(x_1), \dots, \varphi_n(x_n)).$$

Let $x = (x_1, \dots, x_n) \in E$. Then $\varphi_1(x_1) = \dots = \varphi_n(x_n) \in A$ and

$$f(x_1, \dots, x_n) = \tilde{f}(\varphi_1(x_1), \dots, \varphi_n(x_n)) = \tilde{g}(\varphi_1(x_1)) = g(h_1(x_1)) = g(x).$$

Thus, g is the diagonal of the mapping f .

In the case of the pseudocompact set E we can reasoning analogously. We consider the set $S = \mathbb{N}^{n-1}$ and obtain continuous mappings $\varphi_i : E_i \rightarrow \mathbb{R}^S$. Then the set $A = \varphi_1(E_1) = \cdots = \varphi_n(E_n)$ is a functionally closed set as a pseudocompact subsets in the metrizable space Z . \square

In the case of $X_1 = X_2 = \cdots = X_n = X$ $E = \Delta_n$ we obtain the following result.

Theorem 3.2. *Let X be a topological space and $g : X \rightarrow \mathbb{R}$ be a function of $(n-1)$ -th Baire class. Then there exists a separately continuous function $f : X^n \rightarrow \mathbb{R}$ with the diagonal g .*

4. FUNCTIONS ON THE PRODUCT OF COMPACTS

Theorem 4.1. *Let X_1, \dots, X_n be a compacts, E be a closed projectively injective set in $X_1 \times \cdots \times X_n$ and $g : E \rightarrow \mathbb{R}$ be a function of $(n-1)$ -th Baire class. Then there exists a separately continuous function $f : X_1 \times \cdots \times X_n \rightarrow \mathbb{R}$ for which $f|_E = g$.*

Proof. Since the set E is projectively injective, the projective mappings defined on the compact set E and valued in X_1, X_2, \dots, X_n are continuous injective mappings. Thus, they are homeomorphic embeddings. It remains to use Theorem 3.1. \square

Theorem 4.2. *Let X_1, \dots, X_n be a locally compact spaces such that $X = X_1 \times \cdots \times X_n$ be a paracompact, $E \subseteq X$ be a closed locally projectively injective set and $g : E \rightarrow \mathbb{R}$ be a function of $(n-1)$ -th Baire class. Then there exists a separately continuous function $f : X_1 \times \cdots \times X_n \rightarrow \mathbb{R}$ with $f|_E = g$.*

Proof. For every point $p = (x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ we choose open neighborhoods $U_p^{(1)}, \dots, U_p^{(n)}$ of points x_1, \dots, x_n in the spaces X_1, \dots, X_n respectively such that the closures

$$X_p^{(1)} = \overline{U_p^{(1)}}, \dots, X_p^{(n)} = \overline{U_p^{(n)}}$$

are compacts and the set

$$E_p = E \cap (X_p^{(1)} \times \cdots \times X_p^{(n)})$$

is projectively injective. According to Theorem 4.1, for every $p \in X$ there exists a separately continuous function $f_p : X_p^{(1)} \times \cdots \times X_p^{(n)} \rightarrow \mathbb{R}$ with $f_p|_{E_p} = g|_{E_p}$. Since the space X is a paracompact, there exists a partition of the unit $(\varphi_i : i \in I)$ on X which subordinated to the cover $(W_p = U_p^{(1)} \times \cdots \times U_p^{(n)} : p \in X)$ of the space X [17, p.447]. For every $i \in I$ we choose $p_i \in X$ such that $\text{supp} \varphi_i \subseteq W_{p_i}$ and put

$$g_i(x) = \begin{cases} f_{p_i}(x), & \text{if } x \in W_{p_i}, \\ 0, & \text{if } x \notin W_{p_i}. \end{cases}$$

Note that the function $\varphi_i \cdot g_i$ are separately continuous on X and $(\varphi_i g_i)|_E = (\varphi_i|_E)g$. Then the function

$$f = \sum_{i \in I} \varphi_i g_i$$

is a desired function. \square

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